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Computing Geodetic Coordinates in Space

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Nomenclature

a	= equatorial radius of the planet, m
b	= polar radius of the planet, m
$\mathcal{C}$	= section of the planet's surface by a plane containing the polar axis and $\boldsymbol{X}$
$\boldsymbol{C}(\alpha)$	= point on $\mathcal{C}$ with geodetic latitude α
e	= eccentricity of the planet, $e = \sqrt{(1 - \sigma^2)}$ (dimensionless)
$H(\alpha)$	= distance from $\boldsymbol{X}$ to $\boldsymbol{C}(\alpha)$, m
h	= geodetic altitude of $\boldsymbol{X}$, m
$\boldsymbol{N}$	= outward unit normal vector at $\boldsymbol{P}$
$\boldsymbol{O}$	= center of the oblate spheroidal planet
$\boldsymbol{P}$	= intersection of the planet's surface and the line through $\boldsymbol{O}$ and $\boldsymbol{X}$
$\boldsymbol{Q}$	= intersection of the planet's equator and the line through $\boldsymbol{P}$ and $\boldsymbol{X}$
$\boldsymbol{X}$	= point in space whose geodetic coordinates are to be computed
(x, y, z)	= Cartesian (geocentric) coordinates of $\boldsymbol{X}$, m
$\beta_e(\alpha)$	= auxiliary function, $\sqrt{\{1 - e^2[\sin(\alpha)]^2\}}$ (dimensionless)
$\theta(\alpha)$	= angle between $\boldsymbol{C}'(\alpha)$ and $\boldsymbol{C}(\alpha) - \boldsymbol{X}$, deg
κ	= contraction constant (dimensionless)
λ	= geodetic latitude of $\boldsymbol{X}$, deg
μ	= geocentric latitude of $\boldsymbol{X}$, that is, polar angle of the Cartesian point (ρ, z), deg
ρ	= polar radius of the Cartesian point (x, y), $\sqrt{(x^2 + y^2)}$, m
σ	= ratio of the polar and equatorial radii, $\sigma = b/a$ (dimensionless)
ϕ	= geodetic latitude of $\boldsymbol{P}$, deg
φ	= longitude of $\boldsymbol{X}$, that is, polar angle of the Cartesian point (x, y), deg

Subscripts

d	= negative altitudes (deep), $h < 0$
h	= high altitudes (high), $a\sigma^2/\beta_e(\lambda) < h$
ℓ	= low altitudes (low), $0 \leq h \leq a\sigma^2/\beta_e(\lambda)$
$0, 1, n, n+1$	= number of iterations

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Introduction

THIS Note presents a provably accurate algorithm to compute the geodetic latitude and geodetic altitude of a point in space relative to an oblate spheroid (a planet), given the geocentric position of the point relative to the spheroid. This computation is often necessary for navigation and tracking of aircraft, space vehicles, or other objects. The measurements yield the geocentric Cartesian coordinates (x, y, z) of the target $\boldsymbol{X}$:

$$\boldsymbol{X} = \begin{pmatrix} \rho \\ z \end{pmatrix} = \begin{Bmatrix} [a/\beta_e(\lambda) + h]\cos(\lambda) \\ [a\sigma^2/\beta_e(\lambda) + h]\sin(\lambda) \end{Bmatrix} \tag{1}$$

Equation (1) relates the cylindrical coordinates (ρ, z) to the geodetic coordinates (λ, h) (Ref. 1) (Fig. 1). The problem consists of computing the target's geodetic altitute h and geodetic latitude λ given x, y, z. This problem can be solved in closed form (contrary to Deprit and Deprit-Bartholome[2]) because it reduces to solving a quartic equation. However, a closed-form algebraic solution of the quartic equation is impractical for numerical computations for three reasons. First, it requires the computation of a complex cube root, which itself involves a numerical approximation. Second, algebraic solutions contain subtractions that can result in catastrophic cancellations of significant digits. Finally, because of the complexity of the algebraic solutions, no practical upper bounds on the effects of rounding errors, overflow, and underflow appear to exist. The literature contains various numerical approximations for the geodetic coordinates,[3] but apparently does not provide bounds on the errors in the presence of floating-point arithmetic or other perturbations, nor bounds on the number of iterations necessary to achieve a specified accuracy. In one instance, an algorithm[4] published in this journal attempts a division by zero above the poles and near the poles calls for divisions by small numbers that would amplify previous rounding or measurement errors.

In contrast, the algorithm presented here begins with a geocentric approximation and refines it through one iteration of a contracting map. The method is accurate to two-millionths of a degree for

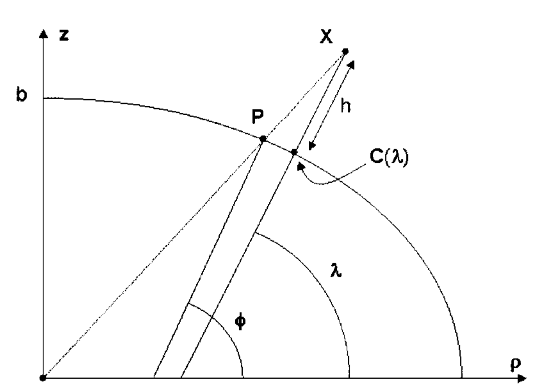

Fig. 1 Problem: given ρ and z for X, compute h and λ.

the geodetic latitude and 1 mm for the geodetic altitude. Further iterations would yield still greater accuracy.

Earth Geodetic Coordinates of Close Objects

The main algorithm is for Earth and for satellites whose orbits are at an altitude of the order of one Earth polar radius or less. Because h is unknown, the inequalities $0 \le h \le b$ cannot be tested, but the quantity $a\sigma^2\beta_e(\lambda)$ is approximately the polar radius b. Moreover, $0 \le h \le a\sigma^2\beta_e(\lambda)$ is algebraically equivalent to

$$(\rho/a)^2 + (z/b)^2 \ge 1 \ge \{\rho^2/[(1+\sigma^2)a]^2\} + z^2/4b^2 \tag{2}$$

which can be calculated and tested from the data. The three-step algorithm comprises a single pass of an iterative method. At each pass the method computes an altitude estimate, estimates of the cosine and sine of latitude, and an intermediate quantity $\beta_e(\lambda)$. The inputs are the geocentric coordinates (x, y, z) of $\boldsymbol{X}$.

Three-Step Algorithm

Step 1

Set $n = 0$. Compute $\rho = \sqrt{(x^2 + y^2)}$ and the initial altitude estimate:

$$h_0 = \left[1 - 1/\sqrt{(\rho/a)^2 + (z/b)^2}\right]\sqrt{\rho^2 + z^2} \tag{3}$$

Step 2

Compute the cosine, $u_0 = \cos(\phi)$, and sine, $v_0 = \sin(\phi)$, of the initial latitude estimate $\lambda_0 = \phi$ and the initial value w_0 of $\beta_e(\lambda_0)$:

$$u_0 = \frac{\sigma^2\rho}{\sqrt{\sigma^4\rho^2 + z^2}}, \qquad v_0 = \frac{z}{\sqrt{\sigma^4\rho^2 + z^2}}, \qquad w_0 = \sqrt{1 - e^2 v_0^2} \tag{4}$$

Step 3

Compute the cosine, u_{n+1}, and sine, v_{n+1}, of the improved latitude estimate λ_{n+1}, the corresponding value w_{n+1} of β_e, and the improved altitude estimate h_{n+1}:

$$u_{n+1} = \frac{\left[a\sigma^2 + h_n w_n\right]\rho}{\sqrt{\left[a\sigma^2 + h_n w_n\right]^2\rho^2 + [a + h_n w_n]^2 z^2}}$$

$$v_{n+1} = \frac{[a + h_n w_n]z}{\sqrt{\left[a\sigma^2 + h_n w_n\right]^2\rho^2 + [a + h_n w_n]^2 z^2}}$$

$$w_{n+1} = \sqrt{1 - e^2 v_{n+1}^2}$$

$$h_{n+1} = \sqrt{\left(\rho - \frac{a u_{n+1}}{w_{n+1}}\right)^2 + \left(z - \frac{a\sigma^2 v_{n+1}}{w_{n+1}}\right)^2} \tag{5}$$

Final Step: Step 4

Compute $\lambda_{n+1} = \arctan(v_{n+1}/u_{n+1})$ with a standard algorithm that is stable near ± 90 deg.

Accuracy of the Three-Step Algorithm

With a single pass through the algorithm, $n = 0$ and the final estimates are $\lambda_{n+1} = \lambda_1$ and $h_{n+1} = h_1$. Tracking the compounding of all of the intermediate rounding errors from the Institute of Electrical and Electronics Engineers (IEEE) double precision binary floating-point arithmetic consists of straightforward but lengthy algebraic inequalities; the general method and many examples relevant to our algorithm are presented in the literature.[5] The result shows that the computed values of λ_1 and h_1 have the following accuracy, valid for all nonnegative altitudes:

$$|\lambda - \lambda_1| < \begin{cases} 2.2 \times 10^{-6}\ \text{deg} & \text{for Earth} \\ 8.5 \times 10^{-2}\ \text{deg} & \text{for Saturn} \end{cases} \tag{6}$$

$$|h - h_1| < \begin{cases} 8.2 \times 10^{-4}\ \text{m} & \text{for Earth} \\ 1.2 \times 10^{+4}\ \text{m} & \text{for Saturn} \end{cases} \tag{7}$$

Subsequent iterations converge with the following accuracy:

$$|\lambda - \lambda_{n+1}| \le |\lambda - \lambda_1| \times \kappa_\ell^n \qquad |h - h_{n+1}| \le |\lambda - \lambda_{n+1}| \times a/\sigma \tag{8}$$

with the dimensionless contraction constant

$$\kappa_\ell = \left\{1 + \left(\sigma + \frac{e^2}{2\sigma^3}\right)\frac{e^2}{1+\sigma}\right\}\frac{e^2}{2\sigma^2} < \begin{cases} 3.4 \times 10^{-3} & \text{for Earth} \\ 1.5 \times 10^{-1} & \text{for Saturn} \end{cases} \tag{9}$$

The realization of such a greater accuracy would require computers with more binary digits than IEEE double precision. Therefore, we have not carried out the analysis of rounding errors past the first iteration.

Accuracy of the Initial Values

In this section we compute bounds on the errors $|h - h_0|$ and $|\lambda - \lambda_0|$. We assume that $0 \le h \le a\sigma^2\beta_e(\lambda)$ and that $\lambda < 90$ deg. (The validity of the algorithm when $\lambda = 90$ deg is obvious.) The derivation of Eqs. (3) and (4) for h_0, u_0, and v_0 appears in textbooks,[1] and so we omit the details. For λ_0 we use ϕ, the geodetic latitude of $\boldsymbol{P}$. From the definition of $\boldsymbol{P}$ in the Nomenclature we obtain

$$\boldsymbol{P} = \frac{1}{\sqrt{(\rho/a)^2 + (z/b)^2}}\begin{pmatrix}\rho \\ z\end{pmatrix} \tag{10}$$

The point on $\mathcal{C}$ closest to $\boldsymbol{X}$ is $\boldsymbol{C}(\lambda)$. The unit normal vector at $\boldsymbol{C}(\lambda)$ is $[\cos(\lambda), \sin(\lambda)]$, and

$$\boldsymbol{C}(\lambda) = \frac{a}{\beta_e(\lambda)}\begin{pmatrix}\cos(\lambda) \\ \sigma^2\sin(\lambda)\end{pmatrix} \tag{11}$$

From Eq. (1) it follows that

$$\tan(\lambda) = \frac{\sin(\lambda)}{\cos(\lambda)} = \frac{1 + [h\beta_e(\lambda)/a]}{\sigma^2 + [h\beta_e(\lambda)/a]}\cdot\frac{z}{\rho} \tag{12}$$

or

$$\lambda = \arctan\left\{\frac{1 + [h\beta_e(\lambda)/a]}{\sigma^2 + [h\beta_e(\lambda)/a]}\cdot\frac{z}{\rho}\right\} \tag{13}$$

Now $\boldsymbol{P} = c\boldsymbol{X}$ for a constant c, and $\boldsymbol{P} = \boldsymbol{C}(\phi) = [a/\beta_e(\phi)]\ [\cos(\phi), \sigma^2\sin(\phi)]$. Therefore, $\tan(\phi) = z/\sigma^2\rho$. For $-\sigma^2 < t$ we write

$$R(t) = (1+t)/(\sigma^2 + t)\cdot z/\rho \tag{14}$$

so that $\lambda = \arctan\{R[h\beta_e(\lambda)/a]\}$ and $\phi = \arctan[R(0)]$. By the mean-value theorem for derivatives applied to the function arctan (and the fact that R is strictly decreasing), if $-\sigma^2 < t_2 \le t_1$ then there is a number t_3 between t_2 and t_1 such that

$$\arctan[R(t_2)] - \arctan[R(t_1)] = [R(t_2) - R(t_1)]\arctan'[R(t_3)]$$
$$= [R(t_2) - R(t_1)]\left\{1/\left(1 + [R(t_3)]^2\right)\right\} \tag{15}$$

Because R decreases and $-\sigma^2 < t_2 \le t_3 \le t_1$, we obtain

$$0 \le \arctan[R(t_2)] - \arctan[R(t_1)]$$
$$\le [R(t_2) - R(t_1)]\left\{1/\left(1 + [R(t_1)]^2\right)\right\} \tag{16}$$

or

$$\arctan[R(t_2)] - \arctan[R(t_1)]$$
$$\le \frac{\left(\sigma^2 + t_1\right)(z/\rho)}{\left(\sigma^2 + t_1\right)^2 + (1+t_1)^2(z/\rho)^2}\,\frac{(1-\sigma^2)(t_1 - t_2)}{\left(\sigma^2 + t_2\right)} \tag{17}$$

The first term on the right is of the form $A\zeta/(A^2 + B^2\zeta^2)$ with $\zeta = z/\rho$, $A = (\sigma^2 + t_1)$, and $B = (1 + t_1)$, and calculus shows that its maximum is $1/(2B)$. This maximum means that

$$\arctan[R(t_2)] - \arctan[R(t_1)] \le e^2/\left[2\left(\sigma^2 + t_2\right)\right](t_1 - t_2)/(1 + t_1) \tag{18}$$

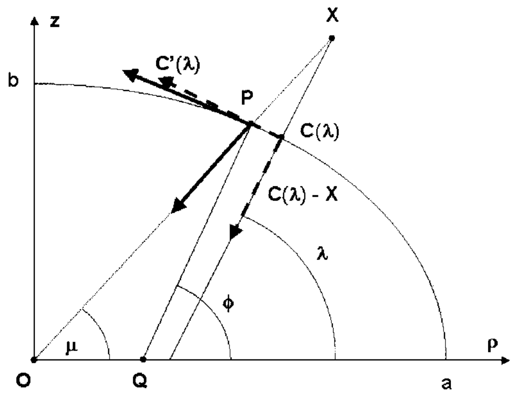

Fig. 2 Auxiliary concepts to facilitate the derivations.

We set $t_1 = h\beta_e(\lambda)/a$ and $t_2 = 0$ and use $0 \le h \le a\sigma^2\beta_e(\lambda)$ to obtain

$$0 \le \lambda_0 - \lambda = \phi - \lambda \le \frac{h\beta_e(\lambda)/a}{1 + h\beta_e(\lambda)/a}\frac{e^2}{2\sigma^2} \le \frac{e^2}{2} \tag{19}$$

We use a similar method to bound the error in h_0. We write $\boldsymbol{C}(\alpha)$ for the point on the ellipse with geodetic latitude α, and $H(\alpha) = \|\boldsymbol{C}(\alpha) - \boldsymbol{X}\|$. The mean-value theorem applied to H gives $|H(\lambda) - H(\lambda_0)| = |(\lambda - \lambda_0)H'(\alpha)|$ for some α with $\lambda < \alpha < \phi$. If $\boldsymbol{X}$ is on the surface, then $h_0 = 0 = h$. Otherwise $H(\alpha) \ne 0$, and (with $\cdot$ denoting the dot product) we find successively that

$$[H(\alpha)]^2 = \|\boldsymbol{C}(\alpha) - \boldsymbol{X}\|^2$$

$$H'(\alpha) = \frac{[\boldsymbol{C}(\alpha) - \boldsymbol{X}] \cdot \boldsymbol{C}'(\alpha)}{\|\boldsymbol{C}(\alpha) - \boldsymbol{X}\|} = \|\boldsymbol{C}'(\alpha)\| \cos[\theta(\alpha)] \tag{20}$$

where $\theta(\alpha)$ denotes the angle between the derivative $\boldsymbol{C}'(\alpha)$ and $\boldsymbol{C}(\alpha) - \boldsymbol{X}$.

Figure 2 shows that $0 < \cos[\theta(\alpha)] < \cos[\theta(\phi)]$ when $\lambda < \alpha < \phi$. From the triangle with vertices $\boldsymbol{O}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ we find that $\theta(\phi) = (\pi/2) - (\phi - \mu)$, where μ is the geocentric latitude of $\boldsymbol{P}$. Hence,

$$\begin{aligned} 0 &\le \cos[\theta(\alpha)] < \cos[\theta(\phi)] \\ &= \sin(\phi - \mu) \\ &< \phi - \mu \\ &= \arctan[(1/\sigma^2)\tan(\mu)] - \arctan[\tan(\mu)] \end{aligned} \tag{21}$$

We use the mean-value theorem again to obtain

$$|\cos[\theta(\alpha)]| < 1/(1 + t^2)[(1/\sigma^2) - 1]\tan(\mu) \tag{22}$$

for some t with $\tan(\mu) < t < \tan(\mu)/\sigma^2$. Then $t/(1 + t^2) \le \frac{1}{2}$ yields

$$|\cos[\theta(\alpha)]| < t/(1 + t^2) \cdot e^2/\sigma^2 \le e^2/2\sigma^2 \tag{23}$$

and then $|\boldsymbol{C}'(\alpha)| = a\sigma^2/[\beta_e(\alpha)]^3 \le a/\sigma$ gives $|H'(\alpha)| \le ae^2/(2\sigma^3)$. Therefore,

$$|h - h_0| = |H(\lambda) - H(\lambda_0)| = |\lambda - \lambda_0||H'(\alpha)| \le ae^4/4\sigma^3 \tag{24}$$

Contracting Map

Compared to methods such as truncated infinite expansions, contracting maps offer the advantage of automatic corrections, with iterations converging to the fixed point regardless of the initial approximation and, hence, regardless of intermediate errors. Newton's method iterates a contracting map, but for the current application it involves canceling subtractions that are difficult to analyze. In contrast, the contracting map presented here has known convergence properties with sharp error bounds. The following derivation demonstrates the method of proof in the case $\lambda \le \lambda_n \le \phi$, which is true for $n = 0$ and $0 \le h \le a\sigma^2\beta_e(\lambda)$, and which yields the error bounds Eqs. (6) and (7) for λ_1 and h_1. We show that the function Λ defined by

$$\begin{aligned} \Lambda(\lambda) &= \arctan\left(\frac{1 + H(\lambda)\beta_e(\lambda)/a}{\sigma^2 + H(\lambda)\beta_e(\lambda)/a} \cdot \frac{z}{\rho}\right) \\ &= \arctan\{R[H(\lambda)\beta_e(\lambda)/a]\} \end{aligned} \tag{25}$$

is a contracting map, that is, that there exists a constant κ with $0 \le \kappa < 1$ such that $|\Lambda(\lambda_n) - \Lambda(\lambda)| \le \kappa|\lambda_n - \lambda|$. With t_1 designating the larger and t_2 the smaller of $H(\lambda_n)\beta_e(\lambda_n)/a$ and $H(\lambda)\beta_e(\lambda)/a$, Eq. (18) gives

$$\begin{aligned} |\Lambda(\lambda_n) - \Lambda(\lambda)| &= |\arctan\{R[H(\lambda_n)\beta_e(\lambda_n)]\} \\ &\quad - \arctan\{R[H(\lambda)\beta_e(\lambda)]\}| \\ &\le \left[e^2 / 2a\left(\sigma^2 + t_2\right)(1 + t_1)\right]|H(\lambda_n)\beta_e(\lambda_n) \\ &\quad - H(\lambda)\beta_e(\lambda)| \\ &= \left[e^2 / 2a\left(\sigma^2 + t_2\right)(1 + t_1)\right]|\lambda_n - \lambda||(H\beta_e)'(\alpha)| \end{aligned} \tag{26}$$

for some α between λ_n and λ, where we have applied the mean-value theorem again to the product $H\beta_e$. Calculus shows that $|\beta_e'| \le (e^2/\sigma^2)\beta_e$, and previous estimates give

$$\begin{aligned} |(H\beta_e)'(\alpha)| &\le |H'(\alpha)\beta_e(\alpha)| + |H(\alpha)\beta_e'(\alpha)| \\ &\le |\boldsymbol{C}'(\alpha)\cos[\theta(\alpha)]\beta_e(\alpha)| + (e^2/\sigma^2)|H(\alpha)\beta_e(\alpha)| \\ &= (e^2/2\sigma^2)\left|a\sigma^2 / [\beta_e(\alpha)]^3\beta_e(\alpha)\right| + e^2/\sigma^2|H(\alpha)\beta_e(\alpha)| \\ &\le (ae^2/\sigma^2)[1 + H(\alpha)\beta_e(\alpha)/a] \\ &\le (ae^2/\sigma^2)[1 + t_1] \end{aligned} \tag{27}$$

Combining these results in Eq. (26) gives

$$|\Lambda(\lambda_n) - \Lambda(\lambda)| \le e^2 / 2\left(\sigma^2 + t_2\right)(e^2/\sigma^2)|\lambda_n - \lambda| < e^4/2\sigma^4|\lambda_n - \lambda| \tag{28}$$

It follows that Λ is a contracting map with contraction constant $\kappa = e^4/2\sigma^4$ provided this constant is less than 1, which holds when $e^2 < 2 - \sqrt{2}$. We have $|\lambda_n - \lambda| \le \kappa^n|\lambda_0 - \lambda|$ and $|h_n - h| \le \gamma\kappa^n|\lambda_0 - \lambda|$, where $\gamma = ae^2/2\sigma^3$. The constants κ (dimensionless) and γ satisfy

$$\kappa < \begin{cases} 2.3 \times 10^{-5} & \text{for} \quad \text{Earth} \\ 3.3 \times 10^{-2} & \text{for} \quad \text{Saturn} \end{cases}$$

$$\gamma < \begin{cases} 2.2 \times 10^{4}\ \text{m/rad} & \text{for} \quad \text{Earth} \\ 8.6 \times 10^{6}\ \text{m/rad} & \text{for} \quad \text{Saturn} \end{cases} \tag{29}$$

For $n = 0$ this inequality means that $|\lambda_1 - \lambda| = |\Lambda(\lambda_0) - \Lambda(\lambda)| \le \kappa \cdot |\lambda_0 - \lambda|$ and $|h_1 - h| \le \gamma \cdot \kappa \cdot |\lambda_0 - \lambda|$, which yields the error bounds claimed in Eqs. (6) and (7).

Initial Values and Error Bounds for Other Cases

For negative altitudes in the range $-a\sigma^2/(600\beta_e) \le h < 0$, which corresponds to the greatest depth (-10^4 m) of all oceans on Earth, we also use $\lambda_0 = \phi$, but the contraction constant κ_d is different. For high altitudes $h > a\sigma^2/\beta_e(\lambda)$, the same initial value $\lambda_0 = \phi$ would work, but we use a different initial value, $\lambda_0 = \mu$, that reduces the error by a factor of two; the contraction constant κ_h is also different (see Table 1). For example, with a satellite on orbit at any altitude higher than one Earth radius, in particular for geosynchronous satellites, the first row of Table 1 shows that from the initial estimate $\lambda_0 = \mu$ one iteration of our algorithm still yields a geodetic altitude h_1 accurate to 1 mm and a geodetic latitude λ_1 accurate to one millionth of a degree.

Table 1 Error bounds for various initial values λ_0 and altitude ranges

λ_0	h[a]	$\lvert\lambda - \lambda_0\rvert$	$\lvert h - h_0\rvert$	$\lvert\lambda - \lambda_1\rvert$	$\lvert h - h_1\rvert$	$\lvert\lambda - \lambda_{n+1}\rvert$	$\lvert h - h_{n+1}\rvert$	κ[b]
μ	$h_\ell < h$	$e^2/4\sigma^2$	$ae^4/8\sigma^5$	$e^6/[16(1+\sigma)\sigma^5]$	$ae^8/[32(1+\sigma)\sigma^8]$	$(e^2/4\sigma^2)\kappa^{n+1}$	$(ae^2/2\sigma^3)(e^2/4\sigma^2)\kappa^{n+1}$	κ_h
ϕ	$0 \le h \le h_\ell$	$e^2/2$	$ae^2/2\sigma^3$	$e^6/[4(1+\sigma^2)\sigma^2]$	$ae^8/[4(1+\sigma^2)\sigma^3]$	$[e^6/4(1+\sigma^2)\sigma^2]\kappa^n$	$(a/\sigma)[e^6/4(1+\sigma^2)\sigma^2]\kappa^n$	κ_ℓ
ϕ	$h_d \le h < 0$	$e^2/1198\sigma$	$ae^2/1198\sigma$	$(e^2/1198\sigma)\kappa$	$(a/\sigma)(e^2/1198\sigma)\kappa$	$(e^2/1198\sigma)\kappa^{n+1}$	$(a/\sigma)(e^2/1198\sigma)\kappa^{n+1}$	κ_d

[a] $h_\ell = a\sigma^2/\beta_e(\lambda)$; $h_d = -a\sigma^2/[600\beta_e(\lambda)]$.

[b] $\kappa_h = \frac{e^4}{4(1+\sigma)\sigma^3} < \begin{cases} 5.7 \times 10^{-6} & \text{for Earth} \\ 7.8 \times 10^{-3} & \text{for Saturn} \end{cases}$; see Eq. (9) for κ_ℓ. $\kappa_d = \frac{e^2}{2}\frac{1 + (1/\sigma)[1 + e^2K/2(1-K)](e^2/1+\sigma)}{(1 - K\sigma^2)\{\sigma^2 - [K\sigma^2 + e^2K/2\sigma(1-K)]\}} < \begin{cases} 3.7 \times 10^{-3} & \text{for Earth} \\ 1.5 \times 10^{-1} & \text{for Saturn} \end{cases}$

Conclusion

From geocentric (Cartesian or cylindrical) coordinate data, our algorithm uses a geocentric initial estimate and one iteration of a contracting map to compute geodetic coordinates with a provable accuracy with IEEE binary arithmetic.

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